Lecture 7: September 18

Ahlfors' lemma. We saw in Lecture 5 that the unit disk is an example of a period domain. Today, we will show that, at least as far as period mappings are concerned, arbitrary period domains behave much like bounded domains in $\mathbb{C}^n$.

Let $\Delta_R = \left\{ t \in \mathbb{C} \;\middle|\; |t| < R \right\}$ denote the open disk of radius $R > 0$. Recall from last time that the Poincaré metric h_{Δ_R} is given by the formula

$$h_{\Delta_R}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \frac{2R^2}{(R^2 - |t|^2)^2},$$

The Poincaré metric has constant sectional curvature -1, which is to say that

$$\frac{\partial}{\partial t}\frac{\partial}{\partial \bar{t}}\left(\log \frac{2R^2}{(R^2 - |t|^2)^2}\right) = -\frac{2R^2}{(R^2 - |t|^2)^2}.$$

The following important result, which is a generalization of the Schwarz-Pick lemma from complex analysis, is known as *Ahlfors' lemma.*

Theorem 7.1. *Let M be a complex manifold with a hermitian metric h_M, and let $f\colon \Delta_R \to M$ be a holomorphic mapping. Suppose that the function*

$$\varphi = h_M\left(f_*\frac{\partial}{\partial t}, f_*\frac{\partial}{\partial t}\right) \in C^\infty(\Delta_R)$$

satisfies the inequality $\frac{\partial}{\partial t}\frac{\partial}{\partial \bar{t}} \log \varphi \geq \varphi$ at all points of Δ_R where φ is nonzero. Then

$$\varphi \leq \frac{2R^2}{(R^2 - |t|^2)^2}$$

on the entire disk Δ_R.

Proof. We are going to prove the inequality

$$\varphi \leq \frac{2r^2}{(r^2 - |t|^2)^2}$$

for every $r < R$; this is enough, because we can then let $r \to R$ to get the result. Define the auxiliary function $u \in C^\infty(\Delta_r)$ by the formula

$$\varphi = u \cdot \frac{2r^2}{(r^2 - |t|^2)^2}.$$

We observe that u goes to zero near the boundary of Δ_r, because φ is bounded on Δ_r, whereas $2r^2/(r^2 - |t|^2)^2$ goes to infinity near the boundary. Therefore u must have a maximum at some interior point $t_0 \in \Delta_r$.

If $u(t_0) = 0$, then both u and φ are identically zero, and the inequality is trivally satisfied. (This happens when the mapping f is constant.) We may therefore assume from now on that $u(t_0) > 0$, hence also $\varphi(t_0) > 0$. Since the function $\log u$ has a maximum at the point t_0, we get

$$0 \geq \frac{1}{4}\left(\frac{\partial}{\partial x}\frac{\partial}{\partial x} \log u + \frac{\partial}{\partial y}\frac{\partial}{\partial y} \log u\right) = \frac{\partial}{\partial t}\frac{\partial}{\partial \bar{t}} \log u$$

for $t = t_0$, where $t = x + iy$. It follows that

$$0 \geq \frac{\partial}{\partial t}\frac{\partial}{\partial \bar{t}} \log u = \frac{\partial}{\partial t}\frac{\partial}{\partial \bar{t}} \log \varphi - \frac{\partial}{\partial t}\frac{\partial}{\partial \bar{t}}\left(\log \frac{2r^2}{(r^2 - |t|^2)^2}\right) \geq \varphi - \frac{2r^2}{(r^2 - |t|^2)^2}$$

for $t = t_0$, which says exactly that $u(t_0) \leq 1$. But u had a maximum at t_0, and therefore $u \leq 1$ on the entire disk Δ_r. $\square$

Ahlfors' lemma says that the length of the tangent vector $f_* \frac{\partial}{\partial t}$, computed with respect to the hermitian metric on M, is at most the length of $\frac{\partial}{\partial t}$, computed with respect to the Poincaré metric. We can integrate this infinitesimal result along curves to obtain the following global version.

Corollary 7.2. *Under the same assumptions as in Theorem 7.1, we have*

$$d_M\big(f(t_1), f(t_2)\big) \leq d_{\Delta_R}(t_1, t_2)$$

for every pair of points $t_1, t_2 \in \Delta_R$.

In other words, the length of the shortest curve connecting the two image points $f(t_1)$ and $f(t_2)$, computed using the metric h_M, is at most the distance of t_1 and t_2 with respect to the Poincaré metric. In that case, one says that the mapping f is "distance decreasing".

Period mappings are distance decreasing. We are now going to prove that period mappings have this property. Because of Ahlfors' lemma, this is a local problem. We will therefore consider the period mapping

$$\Phi \colon \Delta_R \to D$$

of a polarized variation of Hodge structure of weight n on a small disk Δ_R. As usual, we denote the holomorphic vector bundle by $\mathscr{V}$, the connection by ∇, and the Hodge bundles by $F^p\mathscr{V}$. Let V be the space of ∇-flat sections of $\mathscr{V}$ on Δ_R, and $h \colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ the hermitian pairing induced by the polarization on $\mathscr{V}$. The Hodge bundles $F^p\mathscr{V}$ are then subbundles of the trivial bundle $\mathscr{O}_{\Delta_R} \otimes_{\mathbb{C}} V$. The points of the period domain D then correspond to polarized Hodge structures of weight n on V that are polarized by h. We shall use the point $o = \Phi(0)$ as our reference point.

Let h_D be the $G_{\mathbb{R}}$-invariant hermitian metric on D, constructed in Lecture 6. As in Ahlfors' lemma, we introduce the function

$$\varphi = h_D\left(\Phi_* \frac{\partial}{\partial t}, \Phi_* \frac{\partial}{\partial t}\right) \in C^\infty(\Delta_r).$$

Here is the key result.

Theorem 7.3. *There is a constant $\varepsilon > 0$, depending only on D, such that*

$$\frac{\partial}{\partial t}\frac{\partial}{\partial \bar{t}} \log \varphi \geq \varepsilon \cdot \varphi.$$

In fact, we can always arrange that $\varepsilon = 1$, by rescaling the polarization on $\mathscr{V}$ by a factor of ε^{-1}. (This changes the hermitian metric h_D, and hence the function φ, by ε^{-1}, and therefore removes the ε from the right-hand side of the inequality.)

I am going to divide the proof of the theorem into six steps.

Step 1. Since the metric h_D is $G_{\mathbb{R}}$-invariant, all points of D are equivalent, and so it suffices to prove the inequality in the theorem at $t = 0$. Doing out the derivatives, we statement we need to prove is that

$$\varphi(0)\frac{\partial^2 \varphi}{\partial t \partial \bar{t}}(0) - \left|\frac{\partial \varphi}{\partial t}(0)\right|^2 \geq \varepsilon \cdot \varphi(0)^3. \tag{7.4}$$

To check this, we only need the first few terms in the Taylor expansion of φ; to be precise, the coefficients at 1, t, $\bar{t}$, and $|t|^2$. In fact, all the functions that appear are real-analytic (because the period mapping Φ is holomorphic). We will use this idea in a few places, to simplify the computation.

Step 2. To do the computation, we need a good presentation for the period mapping. The reference Hodge structure F_o induces a Hodge structure of weight 0 on $\operatorname{End}(V)$, polarized by the pairing $\operatorname{tr}(AB^*)$. To keep the notation simple, I will drop the subscript o when refering to this Hodge structure. We have

$$T_{\Phi(0)}D \cong \operatorname{End}(V)/F^0 \operatorname{End}(V) \cong \bigoplus_{\ell \leq -1} \operatorname{End}(V)^{\ell,-\ell},$$

and so the holomorphic tangent space to $\check{D}$ at the point $\Phi(0)$ is isomorphic to the space on the right, which is a subspace of $\operatorname{End}(V)$. The exponential mapping

$$\exp\colon \operatorname{End}(V) \to \operatorname{GL}(V), \quad \exp(A) = e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!},$$

therefore restricts to a biholomorphic isomorphism between an open neighborhood of the origin in $\bigoplus_{\ell \leq -1} \operatorname{End}(V)^{\ell,-\ell}$ and an open neighborhood of the point $\Phi(0) \in \check{D}$. This means that on a sufficiently small neighborhood of the origin, we can write the period mapping *uniquely* in the form

$$\Phi(t) = e^{A(t)} \cdot \Phi(0),$$

where the function

$$A\colon \Delta_r \to \bigoplus_{\ell \leq -1} \operatorname{End}(V)^{\ell,-\ell}$$

is holomorphic and satisfies $A(0) = 0$ (and where $r \ll R$ in general). In terms of the Hodge filtrations, this says that

$$F_{\Phi(t)} = e^{A(t)} F_o.$$

By construction, A decomposes into a finite sum

$$A = A_{-1} + A_{-2} + \cdots,$$

where each $A_\ell\colon \Delta_r \to \operatorname{End}(V)^{\ell,-\ell}$ is again holomorphic and $A_\ell(0) = 0$.

Step 3. We need to express the Griffiths transversality condition in terms of $A(t)$. For any $v \in F_o^p$, the function $e^{A(t)}v$ is a holomorphic section of the Hodge bundle $F^p\mathscr{V}$, and so its derivative

$$\nabla_{\frac{\partial}{\partial t}} \left(e^{A(t)} v \right) = \frac{\partial}{\partial t} e^{A(t)} v$$

must be a section of $F^{p-1}\mathscr{V}$, hence take values in $F^{p-1}_{\Phi(t)} = e^{A(t)} F_o^{p-1}$. After moving the exponential factor to the other side, we obtain

$$e^{-A(t)} \frac{\partial}{\partial t} e^{A(t)} \in F^{-1} \operatorname{End}(V).$$

When we expand the exponential series, all the terms that appear on the left-hand side belong to the subalgebra $\bigoplus_{\ell \leq -1} \operatorname{End}(V)^{\ell,-\ell}$, which intersects $F^{-1}\operatorname{End}(V)$ only in the subspace $\operatorname{End}(V)^{-1,1}$. The conclusion is that $e^{-A(t)} \frac{\partial}{\partial t} e^{A(t)}$ has to equal its $(-1,1)$-component, which is easily seen to be $\frac{\partial}{\partial t} A_{-1}(t)$. Equivalently,

$$\frac{\partial}{\partial t} e^{A(t)} = e^{A(t)} \frac{\partial A_{-1}(t)}{\partial t}. \tag{7.5}$$

Since we only care about the first few terms in the Taylor expansion, let me write

$$A(t) \equiv Bt + \frac{1}{2} C t^2 \mod t^3,$$

where $B = B_{-1} + B_{-2} + \cdots$ and $C = C_{-1} + C_{-2} + \cdots$ are the Hodge components of the two operators $B, C \in \operatorname{End}(V)$. We have

$$\begin{aligned} e^{A(t)} &\equiv \mathrm{id} + Bt + \frac{1}{2}(B^2 + C)t^2 \mod t^3, \\ \frac{\partial}{\partial t} e^{A(t)} &\equiv B + (B^2 + C)t \mod t^2, \\ A_{-1}(t) &\equiv B_{-1}t + \frac{1}{2} C_{-1} t^2 \mod t^2, \\ \frac{\partial A_{-1}(t)}{\partial t} &\equiv B_{-1} + C_{-1} t \mod t^2. \end{aligned}$$

and so (7.5) amounts to the condition that

$$B + (B^2 + C)t \equiv (\mathrm{id} + Bt)(B_{-1} + C_{-1}t) \mod t^2,$$

It follows that $B = B_{-1}$ and $C = C_{-1}$ both belong to the subspace $\operatorname{End}(V)^{-1,1}$. This fact will be of crucial importance later on.

Step 4. Now we are ready to start deriving a formula for the function

$$\varphi = h_D \left(\Phi_* \frac{\partial}{\partial t}, \Phi_* \frac{\partial}{\partial t} \right).$$

Under the isomorphism

$$T_{\Phi(t)} D \cong \operatorname{End}(V) / F^0_{\Phi(t)} \operatorname{End}(V) \cong \bigoplus_{\ell \leq -1} \operatorname{End}(V)^{\ell, -\ell}_{\Phi(t)},$$

the tangent vector $\Phi_* \frac{\partial}{\partial t}$ is represented by the image of

$$\frac{\partial}{\partial t} e^{A(t)} \cdot e^{-A(t)} = e^{A(t)} \frac{\partial A_{-1}(t)}{\partial t} e^{-A(t)} \tag{7.6}$$

in the quotient. Note that the right-hand side belongs to $F^{-1}_{\Phi(t)} \operatorname{End}(V)$. In order to compute $\varphi(t)$, we therefore need to do the following: One, find the component of (7.6) in the subspace

$$\operatorname{End}(V)^{-1,1}_{\Phi(t)} = F^{-1}_{\Phi(t)} \operatorname{End}(V) \cap \left(F^0_{\Phi(t)} \operatorname{End}(V) \right)^{\perp},$$

where the $\perp$ is taken with respect to the trace pairing on $\operatorname{End}(V)$. Two, compute its square norm with respect to the inner product on $\operatorname{End}(V)$, which is minus the trace pairing on the subspace $\operatorname{End}(V)^{-1,1}_{\Phi(t)}$.

Here is a fairly simple way to do this. Recall that $F^0_{\Phi(t)} \operatorname{End}(V)$ is the space of endomorphisms that preserve the Hodge filtration $F_{\Phi(t)} = e^{A(t)} F_o$; hence

$$F^0_{\Phi(t)} \operatorname{End}(V) = e^{A(t)} \cdot F^0 \operatorname{End}(V) \cdot e^{-A(t)}.$$

We can therefore write the projection of (7.6) to the subspace $F^0_{\Phi(t)} \operatorname{End}(V)$ as

$$e^{A(t)} P(t) e^{-A(t)},$$

where $P \colon \Delta_r \to F^0 \operatorname{End}(V)$ is a real-analytic function; the fact that $B \in \operatorname{End}(V)^{-1,1}$ says that $P(0) = 0$. Since the Hodge decomposition on $\operatorname{End}(V)$ is orthogonal with respect to the trace pairing, the projection is uniquely determined by the condition that, for every $S \in F^0 \operatorname{End}(V)$,

$$\operatorname{tr} \left(e^{A(t)} \frac{\partial A_{-1}(t)}{\partial t} e^{-A(t)} \left(e^{A(t)} S e^{-A(t)} \right)^* \right) = \operatorname{tr} \left(e^{A(t)} P(t) e^{-A(t)} \left(e^{A(t)} S e^{-A(t)} \right)^* \right).$$

Equivalently, for every $S \in F^0 \operatorname{End}(V)$,

$$\operatorname{tr} \left(e^{A(t)^*} e^{A(t)} \left(\frac{\partial A_{-1}(t)}{\partial t} - P(t) \right) e^{-A(t)} e^{-A(t)^*} S^* \right) = 0. \tag{7.7}$$

Once we have $P(t)$, we can then compute the value of φ according to the formula

$$\varphi(t) = -\operatorname{tr}\left(e^{A(t)}\left(\frac{\partial A_{-1}(t)}{\partial t} - P(t)\right)e^{-A(t)}\right)\left(e^{A(t)}\left(\frac{\partial A_{-1}(t)}{\partial t} - P(t)\right)e^{-A(t)}\right)^*,$$

where $*$ always means the adjoint with respect to the hermitian pairing h.

Step 5. Now let's start doing the computation. Remember that we only care about the four coefficients of φ at 1, t, $\bar{t}$, and $|t|^2$. We shall therefore write $P(t)$ as

$$P(t) = Xt + Y\bar{t} + Z|t|^2 + \cdots,$$

where $X, Y, Z \in F^0 \operatorname{End}(V)$. If we expand (7.7), but only keep those terms that have a chance to contribute to the coefficients at t, $\bar{t}$, and $|t|^2$, we find that

$$\operatorname{tr}\Big(\big(\mathrm{id} + B^*\bar{t}\big)\big(\mathrm{id} + Bt\big)\big(B + Ct - Xt - Y\bar{t} - Z|t|^2\big)\big(\mathrm{id} - Bt\big)\big(\mathrm{id} - B^*\bar{t}\big)S^*\Big) = 0.$$

Now we simply look at the coefficients at t, $\bar{t}$, and $|t|^2$:

(1) From the coefficient at t, we get the condition that

$$\operatorname{tr}(CS^*) = \operatorname{tr}(XS^*).$$

The left-hand side is zero, because $C \in \operatorname{End}(V)^{-1,1}$, whereas $S \in F^0 \operatorname{End}(V)$. Therefore $\operatorname{tr}(XS^*) = 0$ for every $S \in F^0 \operatorname{End}(V)$, and since the trace pairing is a polarization, we get $X = 0$.

(2) From the coefficient at $\bar{t}$, we get the condition that

$$\operatorname{tr}\big((B^*B - BB^*)S^*\big) = \operatorname{tr}\big(BS^*B^* - BB^*S^*\big) = \operatorname{tr}(YS^*).$$

Since $[B^*, B] = B^*B - BB^* \in \operatorname{End}(V)^{0,0}$, it follows that $Y = [B^*, B]$.

(3) From the coefficient at $|t|^2$, we get the condition that

$$\operatorname{tr}\big(CS^*B^* - CB^*S^*\big) = \operatorname{tr}\big(ZS^* + BYS^* - YBS^*\big).$$

Looking at Hodge types, this simplifies to

$$\operatorname{tr}\big((B^*C - CB^*)S^*\big) = \operatorname{tr}(ZS^*),$$

and therefore $Z = [B^*, C]$.

Next, we write out the Taylor expansion of $e^{A(t)}\left(\frac{\partial A_{-1}(t)}{\partial t} - P(t)\right)e^{-A(t)}$. After substituting X, Y, and Z, and collecting terms, this looks like

$$B + C\,t - Y\,\bar{t} - \Big([B^*, C] + [B, Y]\Big)|t|^2 + \cdots$$

where I have kept the notation $Y = [B^*, B]$. Now plug this into the formula for $\varphi(t)$ from above, and collect like terms. To make the result easier to read, let me again write $\langle\rangle$ for the positive definite inner product on $\operatorname{End}(V)$ induced by the trace pairing and the reference Hodge structure; so for example, $\|B\|^2 = -\operatorname{tr}(BB^*)$ since $B \in \operatorname{End}(V)^{-1,1}$, but $\|Y\|^2 = \operatorname{tr}(YY^*)$ since $Y \in \operatorname{End}(V)^{0,0}$. With this notation, the final result is this: the relevant terms in the Taylor expansion of $\varphi(t)$ are

$$\varphi(t) = \|B\|^2 + \langle C, B\rangle t + \langle B, C\rangle \bar{t} + \Big(\|C\|^2 - \|Y\|^2 - 2\operatorname{Re}\big\langle [B, Y], B\big\rangle\Big)|t|^2 + \cdots$$

Step 6. It remains to check the inequality in (7.4). The left-hand side is

$$\begin{aligned}\|B\|^2\Big(\|C\|^2 - \|Y\|^2 - 2\operatorname{Re}\big\langle [B, Y], B\big\rangle\Big) - \big|\langle C, B\rangle\big|^2 \\ \geq -\|B\|^2\Big(\|Y\|^2 + 2\operatorname{Re}\big\langle [B, Y], B\big\rangle\Big),\end{aligned}$$

on account of the Cauchy-Schwarz inequality $\|B\|^2\|C\|^2 \geq |\langle C, B\rangle|^2$.

Lemma 7.8. *We have* $\big\langle [B, Y], B\big\rangle = -\|Y\|^2$.

Proof. Recall that $Y = [B^*, B] = B^*B - BB^*$. Since $B \in \operatorname{End}(V)^{-1,1}$, we have

$$\big\langle [B, Y], B \big\rangle = -\operatorname{tr}\big((BY - YB)B^*\big) = \operatorname{tr}(YBB^*) - \operatorname{tr}(YB^*B) = -\operatorname{tr}(Y^2).$$

But $\operatorname{tr}(Y^2) = \|Y\|^2$ because $Y \in \operatorname{End}(V)^{0,0}$. □

This reduces the proof of Theorem 7.3 to establishing an inequality of the form

$$\|[B^*, B]\| \geq \varepsilon \|B\|^2,$$

where the constant $\varepsilon > 0$ only depends on the period domain D. The key point is to prove that $[B^*, B] = 0$ implies that $B = 0$.

Lemma 7.9. *There is a constant $\varepsilon > 0$ such that at all points $z \in D$, one has*

$$\|[B^*, B]\|_z \geq \varepsilon \|B\|_z^2$$

for every $B \in \operatorname{End}(V)_z^{-1,1}$.

Proof. Let us first prove this at the reference point o. We observed earlier that $-B^*$ is the adjoint of $B \in \operatorname{End}(V)_o^{-1,1}$ with respect to the positive definite inner product $\langle v', v'' \rangle_o = h(C_o v', v'')$ on the vector space V. The condition $[B^*, B] = 0$ implies that B is a normal operator, hence diagonalizable. But B is also nilpotent, and so $B = 0$. By compactness, this proves the desired inequality when $z = o$.

For general $z \in D$, we choose an element $g \in G_{\mathbb{R}}$ such that $z = g \cdot o$. Then

$$\operatorname{End}(V)_z^{-1,1} = g \operatorname{End}(V)_o^{-1,1} g^{-1},$$

and because conjugation by g transforms $\|\|_o$ into $\|\|_z$, the inequality also holds at the point z, with the same constant ε. □

First applications. In combination with Ahlfors' lemma, Theorem 7.3 has the following consequence.

Corollary 7.10. *Let $\Phi\colon \Delta_R \to D$ be the period mapping of a polarized variation of Hodge structure. After rescaling the polarization, if necessary, one has*

$$d_D\big(\Phi(t_1), \Phi(t_2)\big) \leq d_{\Delta_R}(t_1, t_2)$$

for every $t_1, t_2 \in \Delta_R$.

We can use this to show that – just like holomorphic functions from $\mathbb{C}$ into the unit disk – every period mapping on $\mathbb{C}$ must be constant.

Corollary 7.11. *Any period mapping $\Phi\colon \mathbb{C} \to D$ is constant.*

Proof. Let $t \in \mathbb{C}$ be an arbitrary point. For any $R > |t|$, we have

$$d_D\big(\Phi(t), \Phi(0)\big) \leq d_{\Delta_R}(t, 0) = \log \frac{R + |t|}{R - |t|}.$$

But the right-hand side goes to zero as $R \to +\infty$, and so $\Phi(t) = \Phi(0)$. This means that the period mapping is constant. □